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ON GENERALIZED DERIVATIONS OF PRIME RINGS

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Abstract. In this paper, we extend the notion of a generalized derivation F associated with derivation d to two generalized derivations F and G associated with the same derivation d, as a new idea, to obtain the commutativity of prime rings under certain conditions.

1. Introduction

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [9] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently, refined and extended by a number of authors. In [6], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. Furthermore, Bresar and Vukman [5] studied the notions of a $*$-derivation and a Jordan $*$-derivation of R. In this paper, we extend the notion of a generalized derivation F associated with derivation d to two generalized derivations F and G associated with the same derivation d, as a new idea, to obtain the commutativity of prime rings under certain conditions.

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2. Preliminaries

Throughout R will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as usual the commutator, we shall write $[x, y] = xy - yx$, and $x \circ y = xy + yx$.

Also, we make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

Recall that R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $f : R \to R$ is called a *derivation* if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *generalized derivation* if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

3. Generalized derivations associated with same derivation on prime rings

Throughout the paper, F denotes an onto map on a prime ring R.

THEOREM 3.1. *Let R be a semiprime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $[F(x), y] = [x, G(y)]$ for all $x, y \in R$, then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$[F(x), y] = [x, G(y)], \quad \forall\ x, y \in R. \tag{3.1}$$

Replacing y by yx in the relation (1), we obtain

$$[F(x), yx] = [x, G(yx)], \quad \forall\ x, y \in R.$$

This implies that

$$y[F(x), x] + [F(x), y]x = [x, G(y)x + yd(x)]$$

for every $x, y \in R$, and hence

$$y[F(x), x] = [x, y]d(x) + y[x, d(x)], \quad \forall\ x, y \in R. \tag{3.2}$$

Again, replacing y by zy in the relation (2) and using (2), we get $[x, z]yd(x) = 0$ for all $x, y, z \in R$. Replacing z by $d(x)z$, we have $[x, d(x)]zR[x, d(x)]z = (0)$, for all $x, z \in R$, and hence, by semiprimeness, we get $[x, d(x)]z = 0$ for all $x, z \in R$. This can be written as

$[x, d(x)]R[x, d(x)] = (0)$ for all $x \in R$, and hence by semiprimeness, $[x, d(x)] = 0$ for all $x \in R$. Thus $d(R) \subseteq Z(R)$. $\square$

THEOREM 3.2. *Let R be a semiprime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x)x = xG(x)$ for all $x \in R$. Then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$F(x)x = xG(x) \tag{3.3}$$

for all $x \in R$. On linearizing the above relation (3), we obtain

$$F(x)y + F(y)x = xG(y) + yG(x), \quad \forall\, x, y \in R. \tag{3.4}$$

Again, replacing x by xy in the relation (4) and using (4), we get

$$F(x)yy + xd(y)y + F(y)xy = xyG(y) + yG(x)y + yxd(y) \quad \forall x, y \in R.$$

Multiplying by y on the right side of the relation (4), we get

$$F(x)y^2 + F(y)xy = xG(y)y + yG(x)y \quad \forall\, x, y \in R. \tag{3.5}$$

Combining (6) with (5), we have

$$xd(y)y = yxd(y) + x[y, G(y)] \quad \forall\, x, y \in R. \tag{3.6}$$

Now, replacing x by rx in (7), we have

$$rxd(y)y = yrxd(y) + rx[y, G(y)] \quad \forall\, x, y \in R. \tag{3.7}$$

Multiplying the left side of the relation (7) by r, we get

$$rxd(y)y = ryxd(y) + rx[y, G(y)] \quad \forall\, r, x, y \in R. \tag{3.8}$$

From (7) and (8), we get $[y, r]xd(y) = 0$ for all $r, x, y \in R$, and hence $[y, d(y)]xd(y) = 0$ for all $x, y \in R$. That is, $[y, d(y)]R[y, d(y)] = (0)$ for all $y \in R$. Then by the semiprimeness of R, we get $[y, d(y)] = 0$ for all $y \in R$. This implies that $d(R) \subseteq Z(R)$. $\square$

THEOREM 3.3. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x) \circ G(y) = \pm x \circ y$ for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$F(x) \circ G(y) = x \circ y, \forall\, x, y \in R. \tag{3.9}$$

Replacing y by yz in (9), we have $F(x) \circ G(yz) = x \circ yz$ for every $x, y, z \in R$. This means that

$$F(x) \circ (G(y)z + yd(z)) = (x \circ y)z - y[x, z], \forall\, x, y, z \in R,$$

and hence

$$(F(x)\circ G(y)z - G(y)[F(x),z] + (F(x)\circ y)d(z) - y[F(x),d(z)] = (x\circ y)z - y[x,z] \tag{3.10}$$

for all $x, y, z \in R$. Combining (9) and (10), we get

$$-G(y)[F(x),z] + (F(x)\circ y)d(z) - y[F(x),d(z)] + y[x,z] = 0, \forall\ x,y \in R. \tag{3.11}$$

Replacing z by $F(x)$ in (11), we get

$$(F(x)\circ y)d(F(x)) - y[F(x),d(F(x))] + y[x,F(x)] = 0, \forall\ x,y \in R. \tag{3.12}$$

Again, replacing y by ry in (12), we obtain

$$(r(F(x)\circ y) + [F(x),r]y)d(F(x)) - ry[F(x),d(F(x))] + ry[x,F(x)] = 0, \forall\ x,y,r \in R. \tag{3.13}$$

Multiplying by r on left side of (13), we get

$$r(F(x)\circ y)d(F(x)) - ry[F(x),d(F(x))] + ry[x,F(x)] = 0, \forall\ x,y,r \in R. \tag{3.14}$$

From (13) and (14), we obtain

$$[F(x),r]yd(F(x)) = 0, \forall\ x,y,r \in R. \tag{3.15}$$

Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(x)) = 0$ for every $x \in R$. Since F is onto, we get $d = 0$.

□

Using the similar techniques, when $F(x) \circ G(y) = -x \circ y$, for every $x, y \in R$, the following Corollary 3.4 can be proved.

COROLLARY 3.4. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x) \circ G(y) = \pm x \circ y$ for all $x, y \in R$. If $d \neq 0$, then R is commutative.*

THEOREM 3.5. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $(F(x)y + F(y)x) \pm (xG(y) + yG(x)) = 0$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$F(x)y + F(y)x = xG(y) + yG(x), \forall\ x,y \in R. \tag{3.16}$$

Replacing x by xy in the relation (16), we get

$$F(xy)y + F(y)xy = xyG(y) + yG(xy), \forall\ x,y \in R.$$

This implies that

$$(F(x)y + xd(y))y + F(y)xy = xyG(y) + y(G(x)y + yxd(y)), \forall\ x, y \in R. \tag{3.17}$$

Multiplying (16) with y from the right side, we get

$$F(x)y^2 + F(y)xy = xG(y)y + yG(x)y, \forall\ x, y \in R. \tag{3.18}$$

Combining (17) and (18), we get

$$xd(y)y = yxd(y) + x[y, G(y)], \forall\ x, y \in R. \tag{3.19}$$

Replacing x by rx, where $r \in R$, in (19) and combining with the expression obtained by multiplying (19) with r from the left side, we get

$$[y, r]xd(y) = 0, \forall\ x, y, r \in R. \tag{3.20}$$

Now, replacing y by $F(y)$ in (20), we obtain

$$[F(y), r]xd(F(y) = 0, \forall\ x, y, r \in R, \tag{3.21}$$

and thus $[F(y), r]Rd(F(y)) = (0))$ for every $y, r \in R$. Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(y)) = 0$ for every $y \in R$. Since F is onto, we get $d = 0$.

□

Theorem 3.6. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $[F(x), G(y)] = \pm xy$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$[F(x), G(y)] = \pm xy, \forall x, y \in R. \tag{3.22}$$

Replacing y by yz, where $z \in R$, in the relation (22), we get $[F(x), G(yz)] = \pm xyz$ for every $x, y, z \in R$. This implies that

$$[F(x), G(y)z + yd(z)] = \pm xyz$$

for every $x, y, z \in R$, and hence

$$[F(x), G(y)z] + [F(x), yd(z)] = \pm xyz$$

for every $x, y, z \in R$, and so we get, by hypothesis,

$$G(y)[F(x), z]+[F(x), G(y)]z+y[F(x), d(z)]+[F(x), y]d(z) = \pm xyz, \forall\ x, y, z \in R. \tag{3.23}$$

This implies that

$$G(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) = 0, \forall\ x, y, z \in R. \tag{3.24}$$

Replacing z by $F(x)$ in (24), we obtain

$$y[F(x), d(F(x))] + [F(x), y]d(F(x)) = 0, \forall\ x, y \in R. \tag{3.25}$$

Now, replacing y by ty, where $t \in R$, in the equation (25), we get

$$ty[F(x), d(F(x))] + t[F(x), y]d(F(x)) + [F(x), t]yd(F(x)) = 0, \forall\, x, y, t \in R. \tag{3.26}$$

Multiplying the equation (25) by t on left side, we get

$$ty[F(x), d(F(x))] + t[F(x), y]d(F(x)) = 0, \forall\, x, y\, t \in R. \tag{3.27}$$

Combining (26) with (27), we obtain

$$[F(x), t]yd(F(x)) = 0, \forall\, x, y, t \in R. \tag{3.28}$$

That is, $[F(x), t]Rd(F(x)) = (0)$. Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(y)) = 0$ for every $y \in R$. Since F is onto, we get $d = 0$. $\square$

Theorem 3.7. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $[F(x), G(y)] = \pm d(x) \circ y$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$[F(x), G(y)] = d(x) \circ y, \forall x, y \in R. \tag{3.29}$$

Replacing y by yz, where $z \in R$, in the relation (29), we get $[F(x), G(yz)] = d(x) \circ yz$ for every $x, y, z \in R$. This implies that

$$[F(x), G(y)z + yd(z)] = d(x) \circ yz$$

for every $x, y, z \in R$, and hence

$$[F(x), G(y)z] + [F(x), yd(z)] = d(x) \circ yz$$

for every $x, y, z \in R$, and so we get

$$G(y)[F(x), z] + [F(x), G(y)]z + y[F(x), d(z)] + [F(x), y]d(z) = (d(x) \circ y)z - y[d(x), z] \tag{3.30}$$

for every $x, y, z \in R$. Combining (29) with (30), we get

$$G(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) + y[d(x), z] = 0, \forall x, y, z \in R. \tag{3.31}$$

Replacing z by $zF(x)$ in the equation (31), we get

$$\begin{aligned} &G(y)[F(x), z]F(x) + [F(x), y]d(z)F(x) + [F(x), y)]zd(F(x)) + y[F(x), d(z)]F(x) \\ &+ yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + yz[d(x), F(x)] + y[d(x), z]F(x) = 0 \end{aligned} \tag{3.32}$$

for every $x, y, z \in R$. Multiplying the equation (31) by $F(x)$ on right side, we get

(3.33)
$$G(y)[F(x), z]F(x)+[F(x), y]d(z)F(x)+y[F(x), d(z)]F(x)+y[d(x), z]F(x) = 0$$

for every $x, y, z \in R$. From (32) and (33), we obtain

(3.34)
$$[F(x), y]zd(F(x))+yz[F(x), d(F(x))]+y[F(x), z]d(F(x))+yz[d(x), F(x)] = 0$$

for every $x, y, z \in R$. Now, replacing y by ry, where $r \in R$, in (34), we get

$$r[F(x), y]zd(F(x)) + [F(x), r]yzd(F(x)) + ryz[F(x), d(F(x))] + ry[F(x), z]d(F(x))$$

(3.35)
$$+ ryz[d(x), F(x)] = 0$$

for every $x, y, z \in R$. Multiplying the equation (34) by r on left side, we get

(3.36)
$$r[F(x), y]zd(F(x))+ryz[F(x), d(F(x))]+ry[F(x), z]d(F(x))+ryz[d(x), F(x)] = 0$$

for every $x, y, z \in R$. From (35) and (36), we obtain

(3.37)
$$[F(x), r]yzd(F(x)) = 0, \forall x, y, z, r \in R.$$

This implies that $[F(x), r]Rd(F(x)) = (0)$, for every $x, r \in R$. Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(x)) = 0$ for every $x \in R$. Since F is onto, we get $d = 0$. By the same way, if $[F(x), G(y)] = -x \circ y$, for every $x, y \in R$, then also the result holds.

□

THEOREM 3.8. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $[F(x), G(y)] = \pm x \circ y$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

(3.38)
$$[F(x), G(y)] = x \circ y, \forall x, y \in R.$$

Replacing y by yz, where $z \in R$, in the relation (38), we get $[F(x), G(yz)] = x \circ yz$ for every $x, y, z \in R$. This implies that

$$[F(x), G(y)z + yd(z)] = x \circ yz$$

for every $x, y, z \in R$, and hence

$$[F(x), G(y)z] + [F(x), yd(z)] = x \circ yz$$

for every $x, y, z \in R$, and so we get
(3.39)
$$G(y)[F(x), z]+[F(x), G(y)]z+y[F(x), d(z)]+[F(x), y]d(z) = (x\circ y)z-y[x, z]$$

for every $x, y, z \in R$. Combining (38) with (39), we get
(3.40)
$$G(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) + y[x, z] = 0, \forall x, y, z \in R.$$

Replacing z by $zF(x)$ in the equation (40), we get

$$G(y)[F(x), z]F(x) + [F(x), y]d(z)F(x) + [F(x), y)]zd(F(x)) + y[F(x), d(z)]F(x)$$
(3.41)
$$+ yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + yz[x, F(x)] + y[x, z]F(x) = 0$$

for every $x, y, z \in R$. Multiplying the equation (40) by $F(x)$ on right side, we get
(3.42)
$$G(y)[F(x), z]F(x)+[F(x), y]d(z)F(x)+y[F(x), d(z)]F(x)+y[x, z]F(x) = 0$$

for every $x, y, z \in R$. From (41) and (42, we obtain
(3.43)
$$[F(x), y]zd(F(x))+yz[F(x), d(F(x))]+y[F(x), z]d(F(x))+yz[x, F(x)] = 0$$

for every $x, y, z \in R$. Now, replacing y by ry, where $r \in R$, in (43), we get

$$r[F(x), y]zd(F(x)) + [F(x), r]yzd(F(x)) + ryz[F(x), d(F(x))] + ry[F(x), z]d(F(x))$$
(3.44)
$$+ ryz[x, F(x)] = 0$$

for every $x, y, z \in R$. Multiplying the equation (43) by r on left side, we get
(3.45)
$$r[F(x), y]zd(F(x))+ryz[F(x), d(F(x))]+ry[F(x), z]d(F(x))+ryz[x, F(x)] = 0$$

for every $x, y, z \in R$. From (44) and (45), we obtain

(3.46) $$[F(x), r]yzd(F(x)) = 0, \forall x, y, z, r \in R.$$

This implies that $[F(x), r]Rd(F(x)) = (0)$, for every $x, r \in R$. Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(x)) = 0$ for every $x \in R$. Since F is onto, we get $d = 0$. In the same way, if $[F(x), G(y)] = -x \circ y$, for every $x, y \in R$, then also the result holds.

□

THEOREM 3.9. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $[F(x), y)] \pm x \circ G(y) = 0$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. Firstly, by hypothesis, we have

$$[F(x), y] - x \circ G(y) = 0, \forall x, y \in R. \tag{3.47}$$

Replacing y by yx, where $x \in r$, in the relation (47) and using (47), we get

$$y[F(x), x] - (x \circ yd(x)) = 0, \forall x, y \in R, \tag{3.48}$$

and hence

$$y[F(x), x] - (x \circ y)d(x) + y[x, d(x)] = 0 \tag{3.49}$$

for every $x, y \in R$. Replacing y by $F(x)y$ in (49), we get

$$F(x)y[F(x), x] - (x \circ F(x)y)d(x) + F(x)y[x, d(x)] = 0, \forall\ x, y \in R,$$

and so we obtain

$$F(x)y[F(x), x] - F(x)(x \circ y)d(x) - [x, F(x)]yd(x) + F(x)y[x, d(x)] = 0, \forall x, y \in R. \tag{3.50}$$

Multiplying the equation (49) by $F(x)$ on left side, we get

$$F(x)y[F(x), x] - F(x)(x \circ y)d(x) + F(x)y[x, d(x)] = 0, \forall\ x, y \in R. \tag{3.51}$$

From (50) and (51), we get

$$[x, F(x)]yd(x) = 0, \forall x, y \in R. \tag{3.52}$$

This means that $[x, F(x)]Rd(x) = 0$ for all $x \in R$. Since R is prime, we have $F(R) \subseteq Z(R)$ or $d = 0$. By the same way, if $[F(x), y] + x \circ G(y) = 0$, for every $x, y \in R$, then also the result holds.

□

THEOREM 3.10. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x) \circ y \pm x \circ G(y) = 0$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. Firstly, by hypothesis, we have

$$F(x) \circ y - x \circ G(y) = 0, \forall x, y \in R. \tag{3.53}$$

Replacing y by yx, where $x \in R$, in the relation (53) and using (53), we get

$$y[F(x), x] - x \circ yd(x) = 0, \forall x, y \in R, \tag{3.54}$$

and hence

$$y[F(x),x]+(x\circ y)d(x)-y[x,d(x)]=0 \tag{3.55}$$

for every $x,y\in R$. Replacing y by $F(x)y$ in (55), we get

$$F(x)y[F(x),x]+(x\circ F(x)y)d(x)-F(x)y[x,d(x)]=0,\forall\ x,y\in R,$$

and so we obtain

(3.56)

$$F(x)y[F(x),x]+F(x)(x\circ y)d(x)-[x,F(x)]yd(x)-F(x)y[x,d(x)]=0,\forall x,y\in R.$$

Multiplying the equation (55) by $F(x)$ on left side, we get

$$F(x)y[F(x),x]+F(x)(x\circ y)d(x)-F(x)y[x,d(x)]=0,\forall\ x,y\in R. \tag{3.57}$$

From (56) and (57), we get

$$[x,F(x)]yd(x)=0,\forall x,y\in R. \tag{3.58}$$

This means that $[x,F(x)]Rd(x)=0$ for all $x\in R$. Since R is prime, we have $F(R)\subseteq Z(R)$ or $d=0$. By the same way, if $F(x)\circ y+x\circ G(y)=0$, for every $x,y\in R$, then also the result holds.

□

Theorem 3.11. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x)\circ y\pm[x,G(y)]=0$ for all $x,y\in R$. Then either $d=0$ or $F(R)\subseteq Z(R)$.*

Proof. Firstly, by hypothesis, we have

$$F(x)\circ y+[x,G(y)]=0,\forall x,y\in R. \tag{3.59}$$

Replacing y by yx, where $x\in R$, in the relation (59) and using (59), we get

$$y[F(x),x]-[x,y]d(x)-y[x,d(x)]=0,\forall x,y\in R, \tag{3.60}$$

Replacing y by $F(x)y$ in (60), we get

$$F(x)y[F(x),x]-[x,F(x)y]d(x)-F(x)y[x,d(x)]=0,\forall\ x,y\in R,$$

and so we obtain

(3.61)

$$F(x)y[F(x),x]+F(x)[x,y]d(x)-[x,F(x)]yd(x)-F(x)y[x,d(x)]=0,\forall x,y\in R.$$

Multiplying the equation (60) by $F(x)$ on left side, we get

$$F(x)y[F(x),x]-F(x)[x,y]d(x)-F(x)y[x,d(x)]=0,\forall\ x,y\in R. \tag{3.62}$$

From (61) and (62), we get

$$[x,F(x)]yd(x)=0,\forall x,y\in R. \tag{3.63}$$

This means that $[x, F(x)]Rd(x) = 0$ for all $x \in R$. Since R is prime, we have $F(R) \subseteq Z(R)$ or $d = 0$. By the same way, if $F(x) \circ y - [x, G(y)] = 0$, for every $x, y \in R$, then also the result holds.

□

THEOREM 3.12. *Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d such that $F(x) \circ G(y) = \pm xy$ for all $x, y \in R$. Then either $d = 0$ or $F(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have

$$F(x) \circ G(y) = xy, \forall x, y \in R. \tag{3.64}$$

Replacing y by yz in the relation (64), we get $F(x) \circ G(yz) = xyz$ for every $x, y, z \in R$. This implies that

$$F(x) \circ (G(y)z + F(x) \circ yd(z)) = xyz$$

for every $x, y, z \in R$, and hence

$$F(x) \circ G(y)z + F(x) \circ yd(z) = xyz$$

for every $x, y, z \in R$, and so we get

$$G(y)[F(x), z] - (F(x) \circ y)d(z) + y[F(x), d(z)] = 0, \forall\ x, y, z \in R. \tag{3.65}$$

Replacing z by $F(x)$ in (65), we obtain

$$(F(x) \circ y)d(F(x)) + y[F(x), d(F(x))] = 0, \forall\ x, y \in R. \tag{3.66}$$

Now, replacing y by ty in the equation (66), we get

$$t(F(x) \circ y)d(F(x)) + [F(x), t]yd(F(x)) + ty[F(x), d(F(x))] = 0, \forall\ x, y, t \in R. \tag{3.67}$$

Multiplying the equation (66) by t on left side, we get

$$t(F(x) \circ y)d(F(x)) + ty[F(x), d(F(x))] = 0, \forall\ x, y\, t \in R. \tag{3.68}$$

Combining (67) with (68), we obtain

$$[F(x), t]yd(F(x)) = 0, \forall\ x, y, t \in R. \tag{3.69}$$

That is, $[F(x), t]Rd(F(x)) = (0)$. Since R is prime, we get either $F(R) \subseteq Z(R)$ or $d(F(y)) = 0$ for every $y \in R$. Since F is onto, we get $d = 0$.

□

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